

Asymptotic Behavior of the Stock Price Distribution Density and Implied Volatility in Stochastic Volatility Models

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Abstract We study the asymptotic behavior of distribution densities arising in stock price models with stochastic volatility. The main objects of our interest in the present paper are the density of time averages of the squared volatility process and the density of the stock price process in the Stein-Stein and the Heston model. We find explicit formulas for leading terms in asymptotic expansions of these densities and give error estimates. As an application of our results, sharp asymptotic formulas for the implied volatility in the Stein-Stein and the Heston model are obtained.

Keywords Stein-Stein model · Heston model · Mixing distribution density · Stock price · Bessel processes · Ornstein-Uhlenbeck processes · CIR processes · Asymptotic formulas · Implied volatility

1 Introduction

In [11] and [12], we found sharp asymptotic formulas for the distribution density of the stock price process in the Hull-White model. These formulas were used in [13] to characterize the asymptotic behavior of the implied volatility in the Hull-White model. The present paper is a continuation of [11], [12], and [13]. It concerns the asymptotic behavior of various distribution densities arising in the Stein-Stein and the Heston model.

The stochastic differential equations characterizing the stock price process X_t and the volatility process Y_t in the Stein-Stein model have the following form:

$$\begin{cases} dX_t = \mu X_t dt + |Y_t| X_t dW_t \\ dY_t = q(m - Y_t) dt + \sigma dZ_t, \end{cases} \quad (1)$$

where W_t and Z_t are standard Brownian motions. The initial condition for X_t is denoted by x_0 and for Y_t by y_0 . The stochastic volatility model in (1) was introduced and studied in [19]. In this model, the absolute value of an Ornstein-Uhlenbeck process plays the role of the volatility of the stock. We will assume that $\mu \in \mathbb{R}$, $q > 0$, $m \geq 0$, and $\sigma > 0$.

The Heston model was introduced in [14]. In this model, the stock price process X_t and the volatility process Y_t satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + \sqrt{Y_t} X_t dW_t \\ dY_t = (a + bY_t) dt + c\sqrt{Y_t} dZ_t, \end{cases} \quad (2)$$

where W_t and Z_t are standard Brownian motions. We will assume that $\mu \in \mathbb{R}$, $a \geq 0$, $b < 0$, and $c > 0$. The initial conditions for X_t and Y_t are denoted by x_0 and y_0 , respectively. The volatility equation in (2) can be

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rewritten in the mean-reverting form. This gives

$$\begin{cases} dX_t = \mu X_t dt + \sqrt{Y_t} X_t dW_t \\ dY_t = r(m - Y_t) dt + c\sqrt{Y_t} dZ_t, \end{cases}$$

where $r = -b$ and $m = -\frac{a}{b}$. The volatility equation in (2) is uniquely solvable in the strong sense, and the solution Y_t is a positive stochastic process. This process is called a Cox-Ingersoll-Ross process (a CIR-process). This process was studied in [6]. Interesting results concerning the Heston model were obtained in [7].

It will be assumed throughout the present paper that the models described by (1) and (2) are uncorrelated. This means that the Brownian motions W_t and Z_t driving the stock price equation and the volatility equation in (1) and (2) are independent. In the analysis of the probability distribution of the stock price X_t , the mean-square averages of the volatility process over finite time intervals play an important role. For the Stein-Stein model, we set

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t Y_s^2 ds \right\}^{\frac{1}{2}}. \quad (3)$$

Here Y_t satisfies the second equation in (1). For the Heston model, we have

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t Y_s ds \right\}^{\frac{1}{2}}, \quad (4)$$

where Y_t satisfies the second equation in (2). It will be shown below that for every $t > 0$, the probability distribution of the random variable α_t defined by (3) for the model in (1) and by (4) for the model in (2) admits a distribution density m_t (see Lemma 6.3 and Remark 8.1 below). The function m_t is called the mixing distribution density.

The distribution density of the stock price X_t will be denoted by D_t . The existence of the density D_t follows from formula (9). In this paper, we obtain sharp asymptotic formulas for the distribution density D_t in the case of the uncorrelated Stein-Stein and Heston models. Note that in [7] and [19], the behavior of D_t was studied for the Heston model and the Stein-Stein model, respectively, using a rough logarithmic scale in the asymptotic formulas. Moreover, no error estimates were given in these papers. The results established in the present paper are considerably sharper. We find explicit formulas for leading terms in asymptotic expansions of D_t in the Heston and the Stein-Stein model with error estimates. It would be interesting to obtain similar results for correlated models, since such models have more applications in finance. We hope that the methods employed in the present paper may be useful in the study of the correlated case.

We will next quickly overview the structure of the present paper. In Section 2, the main results of the paper (theorems 2.1, 2.2, 2.3, and 2.4) are formulated. They concern the asymptotic behavior of the mixing distribution density m_t and the stock price distribution density D_t in the Heston model and the Stein-Stein model. Section 3 is devoted to applications of our main results. In this section, we obtain sharp asymptotic formulas for the implied volatility in the Heston and the Stein-Stein model. In section 4, we gather several known facts about CIR-processes and Bessel processes. We also formulate a theorem of Pitman and Yor concerning exponential functionals of Bessel processes. This theorem plays an important role in the present paper. In Section 5, we prove an asymptotic inversion theorem for the Laplace transform in a certain class of functions (see Theorem 5.2). This theorem is useful in the study of the asymptotic behavior of the mixing distribution density, since it is often easier to find explicit formulas for the Laplace transform of this density than to characterize the density itself. In sections 6 - 9, we prove theorems 2.1 - 2.4 and describe the constants appearing in these theorems.

2 Asymptotic formulas for distribution densities

The next four theorems are the main results of the present paper. The first two of them provide explicit formulas for the leading term in the asymptotic expansion of the distribution density D_t of the stock price X_t in the Stein-Stein model and the Heston model, while the other two concern the asymptotic behavior of the mixing distribution density m_t in these models.

Theorem 2.1 Let D_t be the stock price distribution density in model (1) with $q \geq 0$, $m \geq 0$, and $\sigma > 0$. Then there exist positive constants B_1 , B_2 , and B_3 such that

$$D_t(x_0 e^{\mu t} x) = B_1 x^{-B_3} e^{B_2 \sqrt{\log x}} (\log x)^{-\frac{1}{2}} \left(1 + O\left((\log x)^{-\frac{1}{4}}\right)\right), \quad x \rightarrow \infty. \quad (5)$$

Theorem 2.2 Let D_t be the stock price distribution density in model (2) with $a \geq 0$, $b \leq 0$, and $c > 0$. Then there exist positive constants A_1 , A_2 , and A_3 such that

$$D_t(x_0 e^{\mu t} x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-\frac{3}{4} + \frac{a}{c^2}} \left(1 + O\left((\log x)^{-\frac{1}{4}}\right)\right), \quad x \rightarrow \infty. \quad (6)$$

Theorem 2.3 Suppose that $q \geq 0$, $m \geq 0$, and $\sigma > 0$ in model (1). Then there exist positive constants E , F , and G such that

$$m_t(y) = E e^{-G y^2} e^{F y} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right), \quad y \rightarrow \infty. \quad (7)$$

Theorem 2.4 Suppose that $a \geq 0$, $b \leq 0$, and $c > 0$ in model (2). Then there exist positive constants A , B , and C such that

$$m_t(y) = A e^{-C y^2} e^{B y} y^{-\frac{1}{2} + \frac{2a}{c^2}} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right), \quad y \rightarrow \infty. \quad (8)$$

It is known that for uncorrelated stochastic volatility models, the asymptotic behavior of the stock price distribution density D_t near zero is determined by the behavior of D_t near infinity. Indeed, we have $D_t(x_0 e^{\mu t} x^{-1}) = x^3 D_t(x_0 e^{\mu t} x)$, $x > 0$. This equality can be derived from the formula

$$D_t(x_0 e^{\mu t} x) = \frac{1}{x_0 e^{\mu t} \sqrt{2\pi t}} x^{-\frac{3}{2}} \int_0^\infty y^{-1} m_t(y) \exp\left\{-\left[\frac{1}{2ty^2} \log^2 x + \frac{ty^2}{8}\right]\right\} dy \quad (9)$$

(see, e.g., Section 4 in [12]). It follows from Theorem 2.2 that the stock price distribution density $D_t(x)$ in the Heston model behaves at infinity roughly as the function x^{-A_3} and at zero as the function x^{A_3-3} . For the Stein-Stein model, Theorem 2.1 implies that $D_t(x)$ behaves at infinity roughly as the function x^{-B_3} and at zero as the function x^{B_3-3} . In [19], the latter fact was established (in part, only heuristically) in the logarithmic scale (the same scale was used in [7]). More precisely, the following definition of the asymptotic equivalence was used in [7] and [19]: Two functions $F(x)$ and $G(x)$ are called asymptotically equivalent as $x \rightarrow \infty$ (or as $x \rightarrow 0$) if $\frac{\log F(x)}{\log G(x)} \rightarrow 1$ as $x \rightarrow \infty$ (or $x \rightarrow 0$). It is not hard to see that the constant γ in [19], which characterizes the decay of the stock price density D_t near infinity (see formula (20) in [19]), coincides with the constant B_3 in (5) (see lemmas 6.6 and 7.3 for the description of the constant B_3). However, there is an error in formula (21) in [19]. It is stated that $D_t(x)$ behaves near zero as the function $x^{-1+\gamma}$. By Theorem 2.1, the correct power function that characterizes the behavior of the density D_t near zero is the function $x^{-3+\gamma}$.

The values of the constants A_3 and B_3 appearing in theorems 2.1 and 2.2 are given by the following formulas:

$$A_3 = A_3(t, b, c) = \frac{3}{2} + \frac{\sqrt{8C+t}}{2\sqrt{t}}, \quad C = C(t, b, c) = \frac{t}{2c^2} \left(b^2 + \frac{4}{t^2} r_{\frac{t|b|}{2}}^2\right),$$

and

$$B_3 = B_3(t, b, c) = \frac{3}{2} + \frac{\sqrt{8G+t}}{2\sqrt{t}}, \quad G = G(t, q, \sigma) = \frac{t}{2\sigma^2} \left(q^2 + \frac{1}{t^2} r_{qt}^2\right)$$

(see lemmas 6.6, 7.3, 8.2, and 9.1 below). Here r_s denotes the smallest positive root of the entire function $z \mapsto z \cos z + s \sin z$. The zeroes of this function are studied in Section 6. Note that A_3 does not depend on a , while B_3 does not depend on m . It is clear that $A_3 > 2$ and $B_3 > 2$. In [11] and [12], we studied the tail behavior of the stock price distribution density D_t in the Hull-White model. It was established that the function $D_t(x)$ behaves at infinity like the function x^{-2} on the power function scale. This is an extremely slow behavior. No uncorrelated stochastic volatility model has the function $D_t(x)$ decaying like $x^{-2+\epsilon}$, $\epsilon > 0$ (see [12]). The tail of the stock price distribution in the Hull-White model is “fatter” than the corresponding tail in the Heston and the Stein-Stein model.

3 Applications. Asymptotic behavior of the implied volatility

The implied volatility in an option pricing model is the volatility in the Black-Scholes model such that the corresponding Black-Scholes price of the option is equal to its price in the model under consideration. In the present paper, we will only consider European call options, and the implied volatility will be studied as a function of the strike price K . Let us denote by V_0 the pricing function for the European call option in the Heston (or the Stein-Stein) model. Then the implied volatility $I(K)$ satisfies $C_{BS}(K, I(K)) = V_0(K)$, where C_{BS} stands for the Black-Scholes pricing function.

In [13], we studied the implied volatility in the Hull-White model. In the present paper, we characterize the asymptotic behavior of the implied volatility in the Stein-Stein and the Heston model. Our asymptotic formulas are sharper than the formulas which can be obtained from more general results due to Lee, Benaim, and Friz (see [15], [2], [3]). We will first consider the model in (2). It will be assumed below that the market price of volatility risk γ is equal to zero (see, e.g., [9] for the definition of γ). Then, under the corresponding martingale measure \mathbb{P}^* , the system of stochastic differential equations in (2) can be rewritten in the following form:

$$\begin{cases} dX_t = rX_t dt + Y_t X_t dW_t^* \\ dY_t = (a + bY_t) dt + c\sqrt{Y_t} dZ_t^*, \end{cases} \quad (10)$$

where W_t^* and Z_t^* are independent standard one-dimensional Brownian motions, and $r > 0$ is a constant interest rate. This means that under the measure \mathbb{P}^* the new system also describes a Heston model. This is the reason why we assumed that $\gamma = 0$.

Let us consider a European call option associated with the stock price model in (10). The price of such an option at $t = 0$ is given by the following formula:

$$V_0(K) = \mathbb{E}^* [e^{-rT} (X_T - K)_+], \quad (11)$$

where T is the expiration date and K is the strike price. Put $D_T(x) = D_T(x; r, \nu, \xi, x_0, y_0)$. Then we have

$$V_0(K) = e^{-rT} \int_K^\infty x D_T(x) dx - e^{-rT} K \int_K^\infty D_T(x) dx. \quad (12)$$

The implied volatility is often considered as a function of the log-strike k , which in our case is related to the strike price K by the formula $k = \log \frac{K}{x_0 e^{rT}}$. In terms of k , the implied volatility is defined as follows: $\hat{I}(k) = I(K)$, $-\infty < k < \infty$, $0 < K < \infty$. For uncorrelated stochastic volatility models, the behavior of the implied volatility near zero is completely determined by how it behaves near infinity. More precisely, it is known that the implied volatility is symmetric in the following sense: $I\left((x_0 e^{rT})^2 K^{-1}\right) = I(K)$ for all $K > 0$ (see, e.g., [12]). It is clear that the previous equality can be formulated in terms of the log-strike k as follows: $\hat{I}(k) = \hat{I}(-k)$, $-\infty < k < \infty$.

Important results concerning the behavior of the implied volatility $\hat{I}(k)$ in a general case were obtained by Lee (see [15]). He characterized the asymptotic behavior of the implied volatility for large strikes in terms of the moments of the stock price process. In [2], the asymptotic behavior of the implied volatility was linked to the tail behavior of the distribution of the stock price process (see also [3]). The reader interested in more aspects of the asymptotic behavior of the implied volatility can consult Chapter 5 in [9].

The following theorem will be established below:

Theorem 3.1 *For the Heston model,*

$$\hat{I}(k) = \beta_1 k^{\frac{1}{2}} + \beta_2 + \beta_3 \frac{\log k}{k^{\frac{1}{2}}} + O\left(\frac{\psi(k)}{k^{\frac{1}{2}}}\right) \quad (13)$$

as $k \rightarrow \infty$, where

$$\begin{aligned} \beta_1 &= \frac{\sqrt{2}}{\sqrt{T}} \left(\sqrt{A_3 - 1} - \sqrt{A_3 - 2} \right), \\ \beta_2 &= \frac{A_2}{\sqrt{2T}} \left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}} \right), \end{aligned}$$

$$\beta_3 = \frac{1}{\sqrt{2T}} \left(\frac{1}{4} - \frac{a}{c^2} \right) \left(\frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}} \right),$$

and ψ can be any positive increasing function on $(0, \infty)$ such that $\lim_{k \rightarrow \infty} \psi(k) = \infty$.

For the Stein-Stein model,

$$\hat{I}(k) = \gamma_1 k^{\frac{1}{2}} + \gamma_2 + O\left(\frac{\psi(k)}{k^{\frac{1}{2}}}\right) \quad (14)$$

as $k \rightarrow \infty$, where

$$\begin{aligned} \gamma_1 &= \frac{\sqrt{2}}{\sqrt{T}} \left(\sqrt{B_3 - 1} - \sqrt{B_3 - 2} \right), \\ \gamma_2 &= \frac{B_2}{\sqrt{2T}} \left(\frac{1}{\sqrt{B_3 - 2}} - \frac{1}{\sqrt{B_3 - 1}} \right), \end{aligned}$$

and the function ψ is as above.

4 CIR processes and Bessel processes

The volatility in model (2) is described by the square root of an CIR-process. It is known that CIR-processes are related to squared Bessel processes. In the present section, we gather several results concerning Bessel processes and CIR-processes. Let $\delta \geq 0$ and $x \geq 0$, and consider the following stochastic differential equation: $dT_t = \delta dt + 2\sqrt{T_t}dZ_t$, $T_0 = x$ a.s. This equation has a unique nonnegative strong solution T_t , which is called the squared δ -dimensional Bessel process started at x . The following notation is often used for the squared Bessel process: $T_t = BESQ_x^\delta(t)$. We refer the reader to [4, 5, 8, 10, 17, 18] for more information on Bessel processes.

The next lemma links Bessel processes and CIR-processes (see, e.g., [10]).

Lemma 4.1 *Let Y_t be a CIR process satisfying the equation $dY_t = (a + bY_t)dt + c\sqrt{Y_t}dZ_t$ with $Y_0 = x$ \mathbb{P} -a.s., and put $T_t = BESQ_x^{\frac{4a}{c^2}}(t)$. Then $Y_t = e^{bt}T\left(\frac{c^2}{4b}(1 - e^{-bt})\right)$.*

Remark 4.2 If $b = 0$, then we have $Y_t = BESQ_x^{\frac{4a}{c^2}}\left(\frac{c^2}{4}t\right)$.

Pitman and Yor proved the following assertion (see [17, 18]):

Theorem 4.3 *Let $\lambda > 0$. Then*

$$\mathbb{E} \left[\exp \left\{ -\frac{\lambda^2}{2} \int_0^t BESQ_x^\delta(u) du \right\} \right] = [\cosh(\lambda t)]^{-\frac{\delta}{2}} \exp \left\{ -\frac{x\lambda}{2} \tanh(\lambda t) \right\}.$$

The next statement can be derived from Theorem 4.3. This statement can be found, e.g., in [4].

Theorem 4.4 *Let $a \geq 0$, $b < 0$, $c > 0$, and let Y_t be a CIR-process in Lemma 4.1 such that $Y_0 = y_0$ a.s. Then for every $\eta > \frac{1}{2}$,*

$$= \exp \left\{ -\frac{abt}{c^2} \right\} \left(\frac{2\eta}{2\eta \cosh(bt\eta) - \sinh(bt\eta)} \right)^{\frac{2a}{c^2}} \exp \left\{ -\frac{by_0(4\eta^2 - 1) \sinh(bt\eta)}{2c^2\eta \cosh(bt\eta) - c^2 \sinh(bt\eta)} \right\}.$$

Theorem 4.4 is equivalent to the following assertion:

Theorem 4.5 *Let $a \geq 0$, $b < 0$, $c > 0$, and let Y_t be a CIR process in Lemma 4.1 such that $Y_0 = y_0$ a.s. Then for every $\lambda > 0$,*

$$\begin{aligned} & \mathbb{E}_{y_0} \left[\exp \left\{ -\lambda \int_0^t Y_s ds \right\} \right] \\ &= \exp \left\{ -\frac{abt}{c^2} \right\} \left(\frac{\sqrt{b^2 + 2c^2\lambda}}{\sqrt{b^2 + 2c^2\lambda} \cosh(\frac{1}{2}t\sqrt{b^2 + 2c^2\lambda}) - b \sinh(\frac{1}{2}t\sqrt{b^2 + 2c^2\lambda})} \right)^{\frac{2a}{c^2}} \\ & \quad \exp \left\{ -\frac{2y_0\lambda \sinh(\frac{1}{2}t\sqrt{b^2 + 2c^2\lambda})}{\sqrt{b^2 + 2c^2\lambda} \cosh(\frac{1}{2}t\sqrt{b^2 + 2c^2\lambda}) - b \sinh(\frac{1}{2}t\sqrt{b^2 + 2c^2\lambda})} \right\}. \end{aligned} \quad (15)$$

Theorem 4.5 characterizes the Laplace transform of the probability distribution of the random variable $\int_0^t Y_s ds$. In the next section, we will study the asymptotic behavior of the inverse Laplace transform in a class of functions which look like the function on the right-hand side of (15).

5 An asymptotic inverse of the Laplace transform

Let us assume that M is a function on $(0, \infty)$ whose Laplace transform is given by the following formula:

$$\int_0^\infty e^{-\lambda y} M(y) dy = I(\lambda), \quad \lambda > 0,$$

where

$$I(\lambda) = \lambda^{\gamma_1} G_1(\lambda)^{\gamma_2} G_2(\lambda) e^{F(\lambda)}$$

with $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. We will next explain what restrictions are imposed on the functions G_1 , G_2 , and F . The function G_2 is analytic in the closed half-plane $\overline{\mathbb{C}}_+ = \{\lambda : \operatorname{Re}(\lambda) \geq 0\}$ and such that $G_2(0) \neq 0$. It is also assumed that the function G_1 is analytic in $\overline{\mathbb{C}}_+$ except for a simple pole at $\lambda = 0$ with residue 1, that is, $G_1(\lambda) = \frac{1}{\lambda} + \tilde{G}(\lambda)$ where \tilde{G} is an analytic function in $\overline{\mathbb{C}}_+$. In addition, we suppose that G_1 is a nowhere vanishing function in $\mathbb{C}_+ = \{\lambda : \operatorname{Re}(\lambda) > 0\}$. Similarly, the function F is analytic in $\overline{\mathbb{C}}_+$ and has a simple pole at $\lambda = 0$ with residue $\alpha > 0$, that is, $F(\lambda) = \frac{\alpha}{\lambda} + \tilde{F}(\lambda)$, where \tilde{F} is an analytic function in $\overline{\mathbb{C}}_+$. We also suppose that

$$\left| G_1(\lambda)^{\gamma_2} G_2(\lambda) e^{F(\lambda)} \right| \leq \exp \{-|\lambda|^\delta\} \quad (16)$$

as $|\lambda| \rightarrow \infty$ in $\overline{\mathbb{C}}_+$, for some $\delta > 0$ (much less is needed about the decay of the function in (16)).

Remark 5.1 We assume that the function G_1 does not vanish in \mathbb{C}_+ in order to justify the existence of the power $G_1^{\gamma_2}$. Here we use the fact that for a nowhere vanishing analytic function f in \mathbb{C}_+ , there exists an analytic function g on \mathbb{C}_+ such that $f(\lambda) = e^{g(\lambda)}$, $\lambda \in \mathbb{C}_+$ (see Theorem 6.2 in [20]).

The next result provides an asymptotic formula for the inverse Laplace transform of the function I .

Theorem 5.2 *Suppose that the functions G_1 , G_2 , and F are such as above. Then the following asymptotic formula holds:*

$$M(y) = \frac{1}{2\sqrt{\pi}} \alpha^{\frac{1}{4} + \frac{\gamma_1 - \gamma_2}{2}} G_2(0) e^{\tilde{F}(0)} y^{-\frac{3}{4} + \frac{\gamma_2 - \gamma_1}{2}} e^{2\sqrt{\alpha}\sqrt{y}} \left(1 + O\left(y^{-\frac{1}{4}}\right) \right) \quad (17)$$

as $y \rightarrow \infty$.

Proof. Using the Laplace transform inversion formula, we see that for every $\varepsilon > 0$ we have $M(y) = \frac{1}{2\pi i} \int_{z=\varepsilon+ir} I(z) e^{yz} dz$. It follows that $M(\alpha y) = \frac{1}{2\pi i} \int_{z=\varepsilon+ir} I(z) e^{\alpha y z} dz$. By (16) and Cauchy's formula, we can deform the contour of integration into a new contour η consisting of the following three parts: the half-line $(-\infty i, -y^{-\frac{1}{2}} i]$, the half-circle Γ in the right half-plane of radius $y^{-\frac{1}{2}}$ centered at 0 (it is

oriented counterclockwise), and finally the half-line $[y^{-\frac{1}{2}}i, \infty i)$. It follows that

$$\begin{aligned} M(\alpha y) &= \frac{1}{2\pi i} \int_{\eta} I(z) e^{\alpha y z} dz = \frac{1}{2\pi} \int_{-\infty}^{-y^{-\frac{1}{2}}} (ir)^{\gamma_1} G_1(ir)^{\gamma_2} G_2(ir) e^{\tilde{F}(ir)} e^{-\frac{i\alpha}{r}} e^{i\alpha y r} dr \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} z^{\gamma_1} G_1(z)^{\gamma_2} G_2(z) e^{\tilde{F}(z)} e^{\frac{\alpha}{z}} e^{\alpha y z} dz + \frac{1}{2\pi} \int_{y^{-\frac{1}{2}}}^{\infty} (ir)^{\gamma_1} G_1(ir)^{\gamma_2} G_2(ir) e^{\tilde{F}(ir)} e^{-\frac{i\alpha}{r}} e^{i\alpha y r} dr \\ &= I_1(y) + I_2(y) + I_3(y). \end{aligned} \quad (18)$$

We will first estimate $I_2(y)$. This will give the main contribution to the asymptotics. By making a substitution $z = y^{-\frac{1}{2}} e^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we see that

$$\begin{aligned} I_2(y) &= \frac{1}{2\pi} y^{-\frac{1+\gamma_1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\theta\gamma_1} G_1\left(y^{-\frac{1}{2}} e^{i\theta}\right)^{\gamma_2} G_2\left(y^{-\frac{1}{2}} e^{i\theta}\right) \\ &\quad \exp\left\{\tilde{F}\left(y^{-\frac{1}{2}} e^{i\theta}\right)\right\} \exp\left\{\alpha\sqrt{y}e^{-i\theta}\right\} \exp\left\{\alpha\sqrt{y}e^{i\theta}\right\} e^{i\theta} d\theta. \end{aligned}$$

Next, taking into account the formula $\sqrt{y}(e^{i\theta} + e^{-i\theta}) = 2\sqrt{y}\cos\theta = 2\sqrt{y} + 2\sqrt{y}(\cos\theta - 1)$, we obtain

$$\begin{aligned} I_2(y) &= \frac{1}{2\pi} y^{-\frac{1+\gamma_1}{2}} e^{2\alpha\sqrt{y}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\theta(1+\gamma_1)} G_1\left(y^{-\frac{1}{2}} e^{i\theta}\right)^{\gamma_2} G_2\left(y^{-\frac{1}{2}} e^{i\theta}\right) \\ &\quad \exp\left\{\tilde{F}\left(y^{-\frac{1}{2}} e^{i\theta}\right)\right\} \exp\{2\alpha\sqrt{y}(\cos\theta - 1)\} d\theta. \end{aligned} \quad (19)$$

It is easy to see that

$$e^{i\theta(1+\gamma_1)} = 1 + O(\theta), \quad (20)$$

$$G_2\left(y^{-\frac{1}{2}} e^{i\theta}\right) - G_2(0) = O\left(y^{-\frac{1}{2}}\right), \quad \exp\left\{\tilde{F}\left(y^{-\frac{1}{2}} e^{i\theta}\right)\right\} - e^{\tilde{F}(0)} = O\left(y^{-\frac{1}{2}}\right) \quad (21)$$

on the contour Γ . Moreover, using (43) and the mean value theorem, we obtain

$$G_1\left(y^{-\frac{1}{2}} e^{i\theta}\right)^{\gamma_2} - (\sqrt{y}e^{-i\theta})^{\gamma_2} = \left[\sqrt{y}e^{-i\theta} + \tilde{G}\left(y^{-\frac{1}{2}} e^{i\theta}\right)\right]^{\gamma_2} - (\sqrt{y}e^{-i\theta})^{\gamma_2} = O\left(y^{\frac{\gamma_2-1}{2}}\right)$$

on Γ . Therefore,

$$G_1\left(y^{-\frac{1}{2}} e^{i\theta}\right)^{\gamma_2} - y^{\frac{\gamma_2}{2}} = O\left(y^{\frac{\gamma_2}{2}}|\theta|\right) + O\left(y^{\frac{\gamma_2-1}{2}}\right) \quad (22)$$

on Γ . It follows from (19), (20), (21), and (22) that

$$\begin{aligned} I_2(y) &= \frac{1}{2\pi} G_2(0) e^{\tilde{F}(0)} y^{-\frac{1+\gamma_1-\gamma_2}{2}} e^{2\alpha\sqrt{y}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\{2\alpha\sqrt{y}(\cos\theta - 1)\} \\ &\quad \left(1 + O\left(y^{-\frac{1}{2}}\right) + O(|\theta|)\right) d\theta. \end{aligned} \quad (23)$$

We will next employ Laplace's method to estimate the integral appearing in (23). Consider the integral $\int_a^b e^{-s\Phi(x)} \psi(x) dx$, where $\Phi \in C^\infty[a, b]$ and $\psi \in C^\infty[a, b]$ (much less is needed from the functions Φ and ψ), and assume that there is an $x_0 \in (a, b)$ such that $\Phi'(x_0) = 0$, and $\Phi(x_0) > 0$ throughout $[a, b]$. Then the following assertion holds:

Theorem 5.3 *Under the above assumptions, with $s > 0$ and $s \rightarrow \infty$,*

$$\int_a^b e^{-s\Phi(x)} \psi(x) dx = e^{-s\Phi(x_0)} \left[\frac{A}{\sqrt{s}} + O\left(\frac{1}{s}\right) \right], \quad (24)$$

where $A = \sqrt{2\pi} \psi(x_0) (\Phi''(x_0))^{-\frac{1}{2}}$.

The proof of Theorem 5.3 can be found, e.g., in [20].

Using (24) with $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$, $\Phi(x) = 1 - \cos x$, $\psi(x) = 1$, $x_0 = 0$, and $s = 2\alpha\sqrt{y}$, we see that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\alpha\sqrt{y}(\cos\theta-1)} d\theta = \frac{\sqrt{\pi}}{\sqrt{\alpha}} y^{-\frac{1}{4}} + O\left(y^{-\frac{1}{2}}\right), \quad y \rightarrow \infty. \quad (25)$$

Similarly

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\theta| e^{2\alpha\sqrt{y}(\cos\theta-1)} d\theta = O\left(y^{-\frac{1}{2}}\right), \quad y \rightarrow \infty. \quad (26)$$

Therefore, (23), (25), and (26) give

$$I_2(y) = \frac{1}{2\sqrt{\pi\alpha}} G_2(0) e^{\tilde{F}(0)} y^{-\frac{3}{4} + \frac{\gamma_2 - \gamma_1}{2}} e^{2\alpha\sqrt{y}} \left(1 + O\left(y^{-\frac{1}{4}}\right)\right), \quad y \rightarrow \infty. \quad (27)$$

Moreover, using (16) we obtain

$$I_1(y) + I_3(y) = O\left(\exp\{-cy^\delta\}\right), \quad y \rightarrow \infty, \quad (28)$$

for some $c > 0$. It follows from (18), (27), and (28) that

$$M(\alpha y) = \frac{1}{2\sqrt{\pi\alpha}} G_2(0) e^{\tilde{F}(0)} y^{-\frac{3}{4} + \frac{\gamma_2 - \gamma_1}{2}} e^{2\alpha\sqrt{y}} \left(1 + O\left(y^{-\frac{1}{4}}\right)\right), \quad y \rightarrow \infty. \quad (29)$$

Now it is not hard to see that (29) implies (17).

This completes the proof of Theorem 5.2.

6 Proof of Theorem 2.4

We will first discuss the properties of the following complex function:

$$\Phi_s(z) = z \cos z + s \sin z, \quad z \in \mathbb{C}, \quad (30)$$

where $s \geq -1$. The next lemma concerns the zeros of Φ_s . A special case of this lemma was stated in [19] without proof and was used in [7] and [19].

Lemma 6.1 *For all $s \geq -1$, the function Φ_s has only real zeros.*

Proof of Lemma 6.1. For every $n \geq 1$, put $P_n(z) = z \prod_{k=1}^n \left(1 - \frac{z^2}{k^2}\right)$. The function P_n is a polynomial of degree $2n + 1$, all of whose roots ($z = k$, $k \in \mathbb{Z}$, $|k| \leq n$) are real. Put $Q_n(z) = z^{-s+1} \frac{d}{dz} (z^s P_n(z)) = sP_n(z) + zP_n'(z)$. Then Q_n is a polynomial of degree $2n + 1$ which vanishes at $z = 0$. Also, by Rolle's theorem, the function $\frac{d}{dz} (z^s P_n(s))$ vanishes at points strictly between k and $k + 1$, $-n \leq k < n$, since the function $z^s P_n(z)$ vanishes at those points. It follows from the previous considerations that $Q_n(z)$ has all its $2n + 1$ roots that are real. But $P_n(z) \rightarrow \frac{\sin \pi z}{\pi}$ by the product formula. Hence, $Q_n(z) \rightarrow \frac{s}{\pi} \sin \pi z + z \cos \pi z$, and the desired conclusion that all the roots are real follows from the Rouché-Hurwitz theorem. The proof above implicitly used the condition $s > -1$ (for otherwise $z^s P_n(z)$ does not vanish at the origin). The result for $s = -1$ can be derived from that for $s > -1$ by a limiting argument.

This completes the proof of Lemma 6.1.

In the present paper, we consider only the case where $s \geq 0$. It is clear that the function Φ_s is odd and satisfies $\Phi_s(0) = 0$.

Definition 6.2 *For $s \geq 0$, the smallest positive zero of the function Φ_s will be denoted by r_s .*

The number r_s plays an important role throughout the paper. It is not hard to see that $r_0 = \frac{\pi}{2}$, and $r_s \uparrow \pi$ as $s \rightarrow \infty$. Moreover, the function $s \mapsto r_s$ is differentiable and increasing on $(0, \infty)$. Indeed, the value of r_s for $0 < s < \infty$ is equal to the first coordinate of the point in \mathbb{R}^2 where the segment $y = -s^{-1}x$, $\frac{\pi}{2} < x < \pi$, intersects the curve $y = \tan x$. In addition, we have $r_s = \phi^{-1}(s)$, $0 < s < \infty$, where $\phi(u) = -u(\tan u)^{-1}$, $\frac{\pi}{2} < u < \pi$. It is also clear that $\sin(r_0) = 1$, $\cos(r_0) = 0$, and $\Phi'_0(r_0) = -\frac{\pi}{2}$. Moreover, if $s > 0$, then

$$\sin(r_s) > 0, \quad \cos(r_s) < 0, \quad \text{and} \quad \Phi'_s(r_s) < 0. \quad (31)$$

By Lemma 6.1, the function ρ_s defined by

$$\rho_s(z) = z \cosh z + s \sinh z = -i\Phi_s(iz) \quad (32)$$

has only imaginary zeros.

The next lemma concerns the mixing distribution densities.

Lemma 6.3 *Suppose that $a \geq 0$, $b < 0$, and $c > 0$ in the Heston model. Then the probability distribution of the random variable α_t in (4) admits a density m_t for every $t > 0$.*

Proof. We will first prove that the probability distribution of the random variable $\int_0^t Y_s ds$ admits a density \overline{m}_t . This will allow us to prove the existence of the mixing distribution m_t , since m_t can be determined from the formula

$$\overline{m}_t(y) = \frac{1}{2\sqrt{ty}} m_t \left(t^{-\frac{1}{2}} y^{\frac{1}{2}} \right). \quad (33)$$

It follows from Theorem 4.5 that

$$\begin{aligned} \mathbb{E}_{y_0} \left[\exp \left\{ -\frac{\lambda}{2c^2} \int_0^t Y_s ds \right\} \right] &= \exp \left\{ -\frac{abt}{c^2} \right\} \left(\frac{\sqrt{b^2 + \lambda}}{\sqrt{b^2 + \lambda} \cosh(\frac{1}{2}t\sqrt{b^2 + \lambda}) - b \sinh(\frac{1}{2}t\sqrt{b^2 + \lambda})} \right)^{\frac{2a}{c^2}} \\ &\exp \left\{ -\frac{y_0 c^{-2} \lambda \sinh(\frac{1}{2}t\sqrt{b^2 + \lambda})}{\sqrt{b^2 + \lambda} \cosh(\frac{1}{2}t\sqrt{b^2 + \lambda}) - b \sinh(\frac{1}{2}t\sqrt{b^2 + \lambda})} \right\}. \end{aligned} \quad (34)$$

Denote by Ψ_1 and Ψ_2 the functions on the right-hand side and the left-hand side of formula (34), respectively, and put

$$u_{b,t} = -4t^{-2} r_{\frac{1}{2}t|b|}^2 \quad (35)$$

where $t > 0$ and $b \leq 0$. It is not hard to see that the function Ψ_1 can be continued analytically from the half-line to the half-plane

$$\mathbb{C}_{b,t} = \{ \lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -b^2 + u_{b,t} \}. \quad (36)$$

This can be shown using the properties of the zeros of the function ρ_s defined by (32) and the fact that $\lambda = 0$ is not a singularity of the function Ψ_1 . On the other hand, the function Ψ_2 is analytic in the half-plane $\mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0 \}$ (use formula (34) with $\lambda > 0$ to prove the finiteness of the derivative of the function Ψ_2 in \mathbb{C}_+). It follows that formula (34) holds for all $\lambda \in \mathbb{C}_{b,t}$.

Since $i\xi \in \mathbb{C}_{b,t}$ for all $\xi \in \mathbb{R}^1$, equation (34) implies the following formula:

$$\int_0^\infty e^{-i\xi x} d\nu_t(x) = \Psi_1(i\xi), \quad \xi \in \mathbb{R}^1, \quad (37)$$

where ν_t stands for the probability distribution of the random variable $\frac{1}{c^2} \int_0^t Y_s ds$.

It is not hard to see that

$$\Psi_1(\lambda) - c_1 \left(\frac{z}{\rho_s(z)} \right)^{\frac{2a}{c^2}} \exp \left\{ -\frac{(c_2 z^2 - c_3) \sinh z}{\rho_s(z)} \right\}$$

where $z = \frac{1}{2}t\sqrt{b^2 + \lambda}$, $s = -b$, ρ_s is defined by (32), and c_1 , c_2 , and c_3 are some positive constants.

Next, we observe that for z lying in a proper sector of the right-hand plane and $z \rightarrow \infty$, we have

$$(c_2 z^2 - c_3) \frac{\sinh z}{\rho_s(z)} = c_2 z + O(1)$$

and

$$\frac{z}{\rho_s(z)} = 2e^{-z} + O(|e^{-2z}|).$$

This implies that

$$|\Psi_1(\lambda)| \leq C_1 \exp \left\{ -C_2 |\lambda|^{\frac{1}{2}} \right\} \quad (38)$$

for $\lambda \in \mathbb{C}_{b,t}$ and $\lambda \rightarrow \infty$.

It follows from (38) that the function $\xi \mapsto |\Psi_1(i\xi)|$ belongs to the space $L^2(\mathbb{R}^1)$. Taking into account (37) and using Plancherel's Theorem, we see that the measure ν_t is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^1 . Therefore, the density \bar{m}_t exists. This implies the existence of the mixing density m_t , and the proof of Lemma 6.3 is thus completed.

We will next prove Theorem 2.4. The following lemma will be used in the proof. In this lemma, we use the notation introduced in (30) and (35).

Lemma 6.4 *Let $a \geq 0$, $b < 0$, $c > 0$, and let Y_t be a CIR process satisfying the equation $dY_t = (a + bY_t)dt + c\sqrt{Y_t}dZ_t$ and such that $Y_0 = y_0$ a.s. Then the following formula holds:*

$$\begin{aligned} \int_0^\infty e^{-\lambda y} y^{-\frac{1}{2}} \exp \left\{ (b^2 - u_{b,t}) y \right\} m_t \left(\frac{\sqrt{2c}}{\sqrt{t}} \sqrt{y} \right) dy &= \frac{\sqrt{2t}}{c} \exp \left\{ -\frac{abt}{c^2} \right\} \left(\frac{it\sqrt{\lambda + u_{b,t}}}{2\Phi_{\frac{1}{2}t|b|} \left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)} \right)^{\frac{2a}{c^2}} \\ &\exp \left\{ -\frac{ity_0(\lambda + u_{b,t} - b^2) \sinh \left(\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)}{2c^2\Phi_{\frac{1}{2}t|b|} \left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)} \right\}, \end{aligned} \quad (39)$$

The functions on the both sides of (39) are analytic in the right half-plane $\mathbb{C}_0 = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$, and the equality in (39) holds for all $\lambda \in \mathbb{C}_0$.

Proof. It was established above that formula (34) holds for all $\lambda \in \mathbb{C}_{b,t}$, where $\mathbb{C}_{b,t}$ is the half-plane defined by (36). It is not hard to see that Lemma 6.4 follows from (34), Lemma 6.3, and (33).

We will next compute the residue λ_0 of the function

$$\begin{aligned} \Lambda(\lambda) &= \frac{\sqrt{\lambda + u_{b,t}}}{\sqrt{\lambda + u_{b,t}} \cosh \left(\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right) + |b| \sinh \left(\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)} \\ &= \frac{it\sqrt{\lambda + u_{b,t}}}{2\Phi_{\frac{1}{2}t|b|} \left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)} \end{aligned} \quad (40)$$

at $\lambda = 0$. Since

$$\Phi'_{\frac{1}{2}t|b|} \left(r_{\frac{1}{2}t|b|} \right) = \left(1 + \frac{1}{2}t|b| \right) \cos \left(r_{\frac{1}{2}t|b|} \right) - r_{\frac{1}{2}t|b|} \sin \left(r_{\frac{1}{2}t|b|} \right), \quad (41)$$

it is not hard to see that

$$\begin{aligned} \lambda_0 &= \lim_{\lambda \rightarrow 0} \lambda \Lambda(\lambda) = -r_{\frac{1}{2}t|b|} \lim_{\lambda \rightarrow 0} \frac{\lambda}{\Phi_{\frac{1}{2}t|b|} \left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}} \right)} \\ &= -8t^{-2} r_{\frac{1}{2}t|b|}^2 \Phi'_{\frac{1}{2}t|b|} \left(r_{\frac{1}{2}t|b|} \right)^{-1} = 8t^{-2} r_{\frac{1}{2}t|b|}^2 \left| \Phi'_{\frac{1}{2}t|b|} \left(r_{\frac{1}{2}t|b|} \right) \right|^{-1} \\ &= 8t^{-2} r_{\frac{1}{2}t|b|}^2 \left| \left(1 + \frac{1}{2}t|b| \right) \cos \left(r_{\frac{1}{2}t|b|} \right) - r_{\frac{1}{2}t|b|} \sin \left(r_{\frac{1}{2}t|b|} \right) \right|^{-1}. \end{aligned} \quad (42)$$

In the proof of (42), we used (35) and (31).

Our next goal is to apply Theorem 5.2 to the Laplace transform in formula (39). The numbers γ_1 and γ_2 and the functions G_1 , G_2 , and F in Theorem 5.2 are chosen as follows: $\gamma_1 = 0$, $\gamma_2 = \frac{2a}{c^2}$, and $G_1(\lambda) = \frac{1}{\lambda_0}\Lambda(\lambda)$, where Λ and λ_0 are defined by (40) and (42), $G_2(\lambda) = \lambda_0^{\frac{2a}{c^2}} \frac{\sqrt{2t}}{c} \exp\left\{-\frac{abt}{c^2}\right\}$, and

$$F(\lambda) = -\frac{ity_0(\lambda + u_{b,t} - b^2) \sinh\left(\frac{1}{2}t\sqrt{\lambda + u_{b,t}}\right)}{2c^2\Phi_{\frac{1}{2}t|b|}\left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}}\right)}.$$

Note that G_1 is a nowhere vanishing function in \mathbb{C}_+ . It follows from (38) that the decay condition in (16) is satisfied. The next lemma provides explicit formulas for the residue α of the function F at $\lambda = 0$ and for the number $\tilde{F}(0)$.

Lemma 6.5 *The following formulas hold:*

$$\alpha = \frac{ty_0\eta_1}{2c^2|\rho_1|} \left(b^2 + 4t^{-2}r_{\frac{t|b|}{2}}^2\right) > 0. \quad (43)$$

and

$$\tilde{F}(0) = \frac{ty_0}{2c^2\rho_1^2} \left[\left(\eta_1 - \left(4t^{-2}r_{\frac{t|b|}{2}}^2 + b^2\right)\eta_2\right)\rho_1 + \left(4t^{-2}r_{\frac{t|b|}{2}}^2 + b^2\right)\rho_2 \right]. \quad (44)$$

The constants η_1 , η_2 , ρ_1 , and ρ_2 in (43) and (44) are given by

$$\eta_1 = \sin\left(r_{\frac{1}{2}t|b|}\right), \quad \eta_2 = -\frac{t^2 \cos\left(r_{\frac{1}{2}t|b|}\right)}{8r_{\frac{1}{2}t|b|}}, \quad (45)$$

$$\rho_1 = \frac{t^2}{8r_{\frac{1}{2}t|b|}} \left[\left(1 + \frac{1}{2}t|b|\right) \cos\left(r_{\frac{1}{2}t|b|}\right) - r_{\frac{1}{2}t|b|} \sin\left(r_{\frac{1}{2}t|b|}\right) \right], \quad (46)$$

and

$$\rho_2 = \frac{t^4}{128} \left[r_{\frac{1}{2}t|b|}^{-3} \left[\left(1 + \frac{1}{2}t|b|\right) \cos\left(r_{\frac{1}{2}t|b|}\right) - r_{\frac{1}{2}t|b|} \sin\left(r_{\frac{1}{2}t|b|}\right) \right] + 2r_{\frac{1}{2}t|b|}^{-2} \sin\left(r_{\frac{1}{2}t|b|}\right) \right]. \quad (47)$$

Proof. We will need the first two coefficients in the power series representation for the function $\lambda F(\lambda)$. It is not hard to see that

$$\sinh\left(\frac{1}{2}t\sqrt{\lambda + u_{b,t}}\right) = i\eta_1 + i\eta_2\lambda + \dots \quad (48)$$

where η_1 and η_2 are defined by (45). Next, using formula (41) and taking into account that

$$\Phi_{\frac{1}{2}t|b|}''\left(r_{\frac{1}{2}t|b|}\right) = -\Phi_{\frac{1}{2}t|b|}\left(r_{\frac{1}{2}t|b|}\right) - 2\sin\left(r_{\frac{1}{2}t|b|}\right) = -2\sin\left(r_{\frac{1}{2}t|b|}\right),$$

we obtain

$$\Phi_{\frac{1}{2}t|b|}\left(i\frac{1}{2}t\sqrt{\lambda + u_{b,t}}\right) = \rho_1\lambda + \rho_2\lambda^2 + \dots \quad (49)$$

where ρ_1 and ρ_2 are defined by (46) and (47). Using (31), we see that $\eta_1 > 0$ and $\eta_2 > 0$. Moreover, (48), (49), and the definition of F imply the following equality:

$$\begin{aligned} \lambda F(\lambda) &= \frac{ty_0}{2c^2} \frac{\left(-4t^{-2}r_{\frac{t|b|}{2}}^2 - b^2 + \lambda\right)(\eta_1 + \eta_2\lambda + \dots)}{\rho_1 + \rho_2\lambda + \dots} \\ &= \frac{ty_0}{2c^2} \frac{\left(-4t^{-2}r_{\frac{t|b|}{2}}^2 - b^2\right)\eta_1 + \left(\eta_1 - \left(4t^{-2}r_{\frac{t|b|}{2}}^2 + b^2\right)\eta_2\right)\lambda + \dots}{\rho_1 + \rho_2\lambda + \dots} \end{aligned} \quad (50)$$

where ρ_1 , ρ_2 , η_1 , and η_2 are defined in (46), (47), and (45).

Now, it is clear that Lemma 6.5 follows from (50).

We will next complete the proof of Theorem 2.4 and compute the constants appearing in it. By applying Theorem 5.2 to the Laplace transform in (39), we see that

$$\begin{aligned} & y^{-\frac{1}{2}} \exp \left\{ (b^2 - u_{b,t}) y \right\} m_t \left(\frac{\sqrt{2}c}{\sqrt{t}} \sqrt{y}; a, b, c, y_0 \right) \\ &= \frac{1}{2\sqrt{\pi}} \alpha^{\frac{1}{4} - \frac{a}{c^2}} \lambda_0^{\frac{2a}{c^2}} \frac{\sqrt{2t}}{c} \exp \left\{ -\frac{abt}{c^2} \right\} e^{\tilde{F}(0)} y^{-\frac{3}{4} + \frac{a}{c^2}} e^{2\sqrt{\alpha}\sqrt{y}} \left(1 + O \left(y^{-\frac{1}{4}} \right) \right) \end{aligned} \quad (51)$$

as $y \rightarrow \infty$. Next, replacing y by $y^2 \frac{t}{2c^2}$ in formula (51), we obtain

$$\begin{aligned} m_t(y; a, b, c, y_0) &= \frac{1}{2\sqrt{\pi}} \alpha^{\frac{1}{4} - \frac{a}{c^2}} \lambda_0^{\frac{2a}{c^2}} \frac{\sqrt{2t}}{c} \exp \left\{ -\frac{abt}{c^2} \right\} e^{\tilde{F}(0)} \left(\frac{t}{2c^2} \right)^{-\frac{1}{4} + \frac{a}{c^2}} \\ & \quad y^{-\frac{1}{2} + \frac{2a}{c^2}} \exp \left\{ \frac{\sqrt{2\alpha t}}{c} y \right\} \exp \left\{ -\frac{t(b^2 - u_{b,t})}{2c^2} y^2 \right\} \left(1 + O \left(y^{-\frac{1}{2}} \right) \right) \end{aligned} \quad (52)$$

as $y \rightarrow \infty$. Now it is clear that (52) implies Theorem 2.4. In addition, the following lemma describes the constants appearing in Theorem 2.4:

Lemma 6.6 *The constants A , B , and C in Theorem 2.4 are given by*

$$\begin{aligned} A &= \frac{1}{\sqrt{\pi}} \left(\frac{t}{2c^2} \right)^{\frac{1}{4} + \frac{a}{c^2}} \alpha^{\frac{1}{4} - \frac{a}{c^2}} \lambda_0^{\frac{2a}{c^2}} \exp \left\{ -\frac{abt}{c^2} \right\} e^{\tilde{F}(0)}, \\ B &= c^{-1} \sqrt{2\alpha t}, \quad \text{and} \quad C = (2c^2)^{-1} t \left(b^2 + 4t^{-2} r_{\frac{t|b|}{2}}^2 \right), \end{aligned}$$

where the numbers $\lambda_0 > 0$, $\alpha > 0$, and $\tilde{F}(0)$ are defined in (42), (43), and (44), respectively.

This completes the proof of Theorem 2.4.

Corollary 6.7 *The following formula holds:*

$$m_t(y, a, 0, c, y_0) = A e^{-Cy^2} e^{By} y^{-\frac{1}{2} + \frac{2a}{c^2}} \left(1 + O \left(y^{-\frac{1}{2}} \right) \right)$$

as $y \rightarrow \infty$, where $A = \frac{2^{\frac{1}{4} + \frac{a}{c^2}} y_0^{\frac{1}{4} - \frac{a}{c^2}}}{c\sqrt{t}} \exp \left\{ \frac{4y_0}{c^2 t} \right\}$, $B = \frac{2\sqrt{2y_0\pi}}{c^2 t}$, and $C = \frac{\pi^2}{2c^2 t}$.

Corollary 6.7 can be derived from Theorem 2.4 and Lemma 6.6 by using the equality $r_0 = \frac{\pi}{2}$ and the fact that for $b = 0$ we have $\lambda_0 = \frac{4\pi}{t^2}$, $\alpha = \frac{4y_0\pi^2}{c^2 t^3}$, and $\tilde{F}(0) = -\frac{3y_0}{c^2 t}$.

7 Proof of Theorem 2.2

We will next formulate a theorem concerning the asymptotic behavior of certain integral operators. This result is a minor modification of Theorem 4.5 established in [12].

Theorem 7.1 *Let A , ζ , and b be positive Borel functions on $[0, \infty)$, and suppose that the following conditions hold:*

1. *The function A is integrable over any finite sub-interval of $[0, \infty)$.*
2. *The function b is bounded and $\lim_{y \rightarrow \infty} b(y) = 0$.*
3. *There exist $y_1 > 0$, $c > 0$, and γ with $0 < \gamma \leq 1$ such that ζ and b are differentiable on $[y_1, \infty)$, and in addition $|\zeta'(y)| \leq cy^{-\gamma}\zeta(y)$ and $|b'(y)| \leq cy^{-\gamma}b(y)$ for all $y \geq y_1$.*
4. *For every $a > 0$, there exists $y_a > 0$ such that $b(y)\zeta(y) \geq \exp \{-ay^4\}$, $y > y_a$.*

5. There exists a real number l such that $A(y) = e^{ly}\zeta(y)(1 + O(b(y)))$ as $y \rightarrow \infty$.

Then, for every fixed $k > 0$,

$$\begin{aligned} & \int_0^\infty A(y) \exp \left\{ - \left(\frac{w^2}{y^2} + k^2 y^2 \right) \right\} dy \\ &= \frac{\sqrt{\pi}}{2k} \exp \left\{ \frac{l^2}{16k^2} \right\} \zeta \left(k^{-\frac{1}{2}} w^{\frac{1}{2}} \right) \exp \left\{ lk^{-\frac{1}{2}} w^{\frac{1}{2}} \right\} e^{-2kw} \left[1 + O \left(w^{-\frac{\gamma}{2}} \right) + O \left(b \left(k^{-\frac{1}{2}} w^{\frac{1}{2}} \right) \right) \right], \quad w \rightarrow \infty. \end{aligned}$$

The only difference between Theorem 7.1 formulated above and Theorem 4.5 in [12] is that in Theorem 7.1 we do not assume the integrability of the function ζ near zero. It is not hard to see that Theorem 7.1 can be derived from Theorem 4.5 if we replace the function $\zeta(y)$ by the function $A(y)e^{-ly}$ near zero.

Let $A(y) = y^{-1}m_t(y)e^{Cy^2}$, $k = \sqrt{C + \frac{l}{8}}$, $l = B$, $\zeta(y) = Ay^{-\frac{3}{2} + \frac{2a}{c^2}}$, and $b(y) = y^{-\frac{1}{2}}$, where the constants A , B , and C are as in Theorem 2.4. Then, it is not hard to see that condition 2 in Theorem 7.1 follows from formula (8). In addition, it is clear that condition 3 with $\gamma = 1$ holds. The next lemma shows that condition 1 in Theorem 7.1 also holds.

Lemma 7.2 For every $s > 0$, $\int_0^s y^{-1}m_t(y)dy < \infty$.

Proof. The function on the right-hand side of (39) is integrable with respect to λ over the interval $(1, \infty)$. Suppose h is any positive function on $[0, \infty)$ such that its Laplace transform has this property. Then $\int_0^s h(y)y^{-1}dy < \infty$ for all $s > 0$. It follows from this fact and (39) that the function $y \mapsto y^{-\frac{3}{2}}m_t \left(\frac{\sqrt{2}c}{\sqrt{t}}\sqrt{y} \right)$ is integrable over any interval of the form $[0, s]$ with $s > 0$. This implies Lemma 7.2.

Next, applying Theorem 7.1 we see that

$$\begin{aligned} & \int_0^\infty y^{-1}m_t(y) \exp \left\{ - \left(\frac{z^2}{y^2} + \frac{ty^2}{8} \right) \right\} dy \\ &= A \frac{\sqrt{\pi}}{2k} \exp \left\{ \frac{B^2}{16k^2} \right\} k^{\frac{3}{4} - \frac{a}{c^2}} z^{-\frac{3}{4} + \frac{a}{c^2}} \exp \left\{ Bk^{-\frac{1}{2}}\sqrt{z} \right\} e^{-2kz} \left(1 + O \left(z^{-\frac{1}{4}} \right) \right) \end{aligned} \quad (53)$$

as $z \rightarrow \infty$. Replacing z by $\frac{\log x}{\sqrt{2t}}$ in formula (53) and taking into account formula (9) and the equality $k = \frac{\sqrt{8C+t}}{2\sqrt{2}}$, we obtain

$$\begin{aligned} D_t(x_0 e^{\mu t} x) &= \frac{A}{x_0 e^{\mu t}} 2^{-\frac{3}{4} + \frac{a}{c^2}} t^{-\frac{1}{8} - \frac{a}{2c^2}} (8C+t)^{-\frac{1}{8} - \frac{a}{2c^2}} \exp \left\{ \frac{B^2}{2(8C+t)} \right\} \\ &(\log x)^{-\frac{3}{4} + \frac{a}{c^2}} \exp \left\{ \frac{B\sqrt{2}}{t^{\frac{1}{4}}(8C+t)^{\frac{1}{4}}} \sqrt{\log x} \right\} x^{-\left(\frac{3}{2} + \frac{\sqrt{8C+t}}{2\sqrt{t}}\right)} \left(1 + O \left((\log x)^{-\frac{1}{4}} \right) \right) \end{aligned} \quad (54)$$

as $x \rightarrow \infty$.

Now it is clear that formula (54) implies Theorem 2.2. Moreover, the following lemma holds:

Lemma 7.3 The constants A_1 , A_2 , and A_3 in Theorem 2.2 are given by

$$\begin{aligned} A_1 &= \frac{A}{x_0 e^{\mu t}} 2^{-\frac{3}{4} + \frac{a}{c^2}} t^{-\frac{1}{8} - \frac{a}{2c^2}} (8C+t)^{-\frac{1}{8} - \frac{a}{2c^2}} \exp \left\{ \frac{B^2}{2(8C+t)} \right\}, \\ A_2 &= \frac{B\sqrt{2}}{t^{\frac{1}{4}}(8C+t)^{\frac{1}{4}}}, \text{ and } A_3 = \frac{3}{2} + \frac{\sqrt{8C+t}}{2\sqrt{t}}, \text{ where } A, B, \text{ and } C \text{ are defined in Lemma 6.6.} \end{aligned}$$

Let $p \in \mathbb{R}$, and denote by l_p the moment of order p of the function D_t , that is, $l_p = \mathbb{E}[X_t^p] = \int_0^\infty x^p D_t(x) dx$. The next result was obtained in [1].

Lemma 7.4 For $a \geq 0$, $b \leq 0$, $c > 0$, and $p \in \mathbb{R}$, the following statement holds: $q_p < \infty$ if and only if $2 - A_3 < p < A_3 - 1$, where the constant A_3 is such as in Theorem 2.2. For $b = 0$, $q_p < \infty$ if and only if

$$\frac{1}{2} - \frac{\sqrt{4\pi^2 + c^2 t^2}}{2ct} < p < \frac{1}{2} + \frac{\sqrt{4\pi^2 + c^2 t^2}}{2ct}.$$

It is not hard to see that Lemma 7.4 and more precise integrability theorems for the distribution of the stock price follow from Theorem 2.2.

8 Proof of Theorem 2.3

Let Y_t be the volatility process in model (1). Then Y_t^2 is a squared Ornstein-Uhlenbeck process. The Laplace transform of the law of the squared Ornstein-Uhlenbeck process was found by Wenocur [21] and by Stein and Stein [19]. Another explicit expression for this Laplace transform is given in the next formula:

$$\begin{aligned} \mathbb{E}_{y_0} \left[\exp \left\{ -\lambda \int_0^t Y_s^2 ds \right\} \right] &= 2\sqrt{t} e^{\frac{qt}{2}} \left(\frac{\sqrt{w}}{\sqrt{w} \cosh(t\sqrt{w}) + q \sinh(t\sqrt{w})} \right)^{\frac{1}{2}} \\ &\exp \left\{ -\frac{y_0^2 \lambda \sinh(t\sqrt{w})}{\sqrt{w} \cosh(t\sqrt{w}) + q \sinh(t\sqrt{w})} \right\} \exp \left\{ -\frac{2mqy_0 \lambda (\cosh(t\sqrt{w}) - 1)}{\sqrt{w} (\sqrt{w} \cosh(t\sqrt{w}) + q \sinh(t\sqrt{w}))} \right\} \\ &\exp \left\{ \frac{m^2 q^2 \lambda (\sinh(t\sqrt{w}) - t\sqrt{w} \cosh(t\sqrt{w}))}{w (\sqrt{w} \cosh(t\sqrt{w}) + q \sinh(t\sqrt{w}))} \right\} \\ &\exp \left\{ \frac{m^2 q^3 \lambda \left(4 \sinh^2 \left(\frac{t\sqrt{w}}{2} \right) - t\sqrt{w} \sinh(t\sqrt{w}) \right)}{w^{\frac{3}{2}} (\sqrt{w} \cosh(t\sqrt{w}) + q \sinh(t\sqrt{w}))} \right\} \end{aligned} \quad (55)$$

where $\lambda > 0$ and $w = q^2 + 2\sigma^2\lambda$. We will next sketch the proof of formula (55). The first step is to replace the symbols δ , θ , k , σ_0 , and λ in formula (8) in [19] by q , m , σ , y_0 , and $t\lambda$, respectively, and take into account the following relations between the notation in [19] and in the present paper: $A = -\frac{q}{\sigma^2}$, $B = \frac{mq}{\sigma^2}$, $C = -\frac{\lambda}{\sigma^2}$, $a = \frac{1}{\sigma^2} \sqrt{q^2 + 2\sigma^2\lambda}$, $b = \frac{q}{\sqrt{q^2 + 2\sigma^2\lambda}}$, and $ak^2t = t\sqrt{q^2 + 2\sigma^2\lambda}$. We also combine the terms $\frac{a-A}{2a^2} a^2 k^2 t$ and $-\frac{1}{2} \log \left\{ \frac{1}{2} \left(\frac{A}{a} + 1 \right) + \frac{1}{2} \left(1 - \frac{A}{a} \right) e^{2ak^2t} \right\}$ in the expression for N in formula (7) in [19], and after somewhat long and tedious computations show that formula (8) in [19] and formula (55) in the present paper are equivalent.

Remark 8.1 Lemma 6.3 states that for the Heston model, there exists the mixing distribution density m_t for every $t > 0$. The same statement holds for the Stein-Stein model. This can be established using formula (55) and reasoning as in the proof of Lemma 6.3.

It follows from Remark 8.1 that

$$\mathbb{E}_{y_0} \left[\exp \left\{ -\lambda \int_0^t Y_s^2 ds \right\} \right] = \int_0^\infty e^{-\lambda t y^2} m_t(y) dy = \frac{1}{2\sqrt{t}} \int_0^\infty e^{-\lambda y} y^{-\frac{1}{2}} m_t \left(t^{-\frac{1}{2}} y^{\frac{1}{2}} \right) dy. \quad (56)$$

We will next obtain a sharp asymptotic formula for the mixing distribution density in the Stein-Stein model using formulas (55), (56), and the methods developed in Section 6.

Recall that for $s \geq 0$, we denoted by r_s the smallest strictly positive zero of the function $\Phi_s(z) = z \cos z + s \sin z$. It is clear that $z \cosh z + s \sinh z = -i\Phi_s(iz)$. For $q > 0$ and $t > 0$, put $v_{q,t} = -\frac{r_{qt}^2}{t^2}$. Now (55) and (56) give

$$\begin{aligned} &\int_0^\infty e^{-\lambda y} y^{-\frac{1}{2}} \exp \left\{ (q^2 - v_{q,t}) y \right\} m_t \left(\frac{\sqrt{2}\sigma\sqrt{y}}{\sqrt{t}} \right) dy \\ &= \frac{\sqrt{2t}}{\sigma} \exp \left\{ \frac{qt}{2} \right\} F_1(\lambda) \exp \{ F_2(\lambda) + F_3(\lambda) + F_4(\lambda) + F_5(\lambda) \}, \end{aligned} \quad (57)$$

where

$$F_1(\lambda) = \left(\frac{it\sqrt{\lambda + v_{q,t}}}{\Phi_{qt}(it\sqrt{\lambda + v_{q,t}})} \right)^{\frac{1}{2}}, \quad F_2(\lambda) = -\frac{iy_0^2 t (\lambda + v_{q,t} - q^2) \sinh(t\sqrt{\lambda + v_{q,t}})}{2\sigma^2 \Phi_{qt}(it\sqrt{\lambda + v_{q,t}})}, \quad (58)$$

$$F_3(\lambda) = -\frac{imqy_0 t (\lambda + v_{q,t} - q^2) [\cosh(t\sqrt{\lambda + v_{q,t}}) - 1]}{\sigma^2 \sqrt{\lambda + v_{q,t}} \Phi_{qt}(it\sqrt{\lambda + v_{q,t}})}, \quad (59)$$

$$F_4(\lambda) = \frac{im^2 q^2 t (\lambda + v_{q,t} - q^2) [\sinh(t\sqrt{\lambda + v_{q,t}}) - t\sqrt{\lambda + v_{q,t}} \cosh(t\sqrt{\lambda + v_{q,t}})]}{2\sigma^2 (\lambda + v_{q,t}) \Phi_{qt}(it\sqrt{\lambda + v_{q,t}})}, \quad (60)$$

and

$$F_5(\lambda) = \frac{im^2 q^3 t (\lambda + v_{q,t} - q^2) \left[4 \sinh^2 \frac{t\sqrt{\lambda+v_{q,t}}}{2} - t\sqrt{\lambda+v_{q,t}} \sinh(t\sqrt{\lambda+v_{q,t}}) \right]}{2\sigma^2 (\lambda + v_{q,t})^{\frac{3}{2}} \Phi_{qt}(it\sqrt{\lambda+v_{q,t}})}. \quad (61)$$

It is not hard to see that the functions F_1, F_2, F_3, F_4 , and F_5 have removable singularities at

$$\lambda = -v_{q,t} = \frac{r_{qt}^2}{t^2}.$$

In addition, these functions are analytic in \mathbb{C}_+ . Let us denote by λ_1 the residue of the function F_1 at $\lambda = 0$. It is not hard to see that

$$\lambda_1 = \frac{2r_{qt}^2}{t^2 |(1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt})|}. \quad (62)$$

Our next goal is to apply Theorem 5.2 to (57). Put

$$\gamma_1 = 0, \quad \gamma_2 = \frac{1}{2}, \quad G_1(\lambda) = \frac{it\sqrt{\lambda+v_{q,t}}}{\Phi_{qt}(it\sqrt{\lambda+v_{q,t}})}, \quad G_2(\lambda) = \frac{\sqrt{2t}}{\sigma} e^{\frac{qt}{2}} \lambda_1^{\frac{1}{2}}, \quad (63)$$

and

$$F(\lambda) = F_2(\lambda) + F_3(\lambda) + F_4(\lambda) + F_5(\lambda). \quad (64)$$

In the sequel, the symbols α_j and $\tilde{F}_j(0)$ will stand for the numbers in the formulation of Theorem 5.2 associated with the function F_j , $2 \leq j \leq 5$. We will next compute these numbers. The following formula will be helpful in the computations:

$$\frac{(v_{q,t} - q^2 + \lambda) \lambda}{\Phi_{qt}(it\sqrt{\lambda+v_{q,t}})} = \frac{(v_{q,t} - q^2) + \lambda}{\zeta_1 + \zeta_2 \lambda + \dots} = \tau_1 + \tau_2 \lambda + \dots \quad (65)$$

where

$$\begin{aligned} \zeta_1 &= \frac{t^2 ((1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt}))}{2r_{qt}}, \\ \zeta_2 &= \frac{t^2 ((1+qt) \cos(r_{qt}) + r_{qt} \sin(r_{qt}))}{8r_{qt}^3}, \\ \tau_1 &= \frac{2(v_{q,t} - q^2) r_{qt}}{t^2 ((1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt}))} > 0, \end{aligned}$$

and

$$\begin{aligned} \tau_2 &= \frac{\zeta_1 - (v_{q,t} - q^2) \zeta_2}{\zeta_1^2} \\ &= \frac{4r_{qt}^2 ((1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt})) - (v_{q,t} - q^2) ((1+qt) \cos(r_{qt}) + r_{qt} \sin(r_{qt}))}{2r_{qt} t^2 ((1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt}))^2}. \end{aligned}$$

Here we use the facts that $v_{q,t} = -\frac{r_{qt}^2}{t^2} < 0$, $\cos(r_{qt}) < 0$, $\sin(r_{qt}) > 0$, $\Phi_{qt}(r_{qt}) = 0$,

$$\Phi'_{qt}(r_{qt}) = (1+qt) \cos(r_{qt}) - r_{qt} \sin(r_{qt}),$$

and

$$\Phi''_{qt}(r_{qt}) = -2 \sin(r_{qt}) - \Phi_{qt}(r_{qt}) = -2 \sin(r_{qt}).$$

We will next employ (58)-(61) to find explicit formulas for the numbers α_j and $\tilde{F}_j(0)$ for $2 \leq j \leq 5$. It follows from (58), (65), and the formula

$$\sinh(t\sqrt{\lambda+v_{q,t}}) = i \sin(r_{qt}) - i \frac{t^2}{2r_{qt}} \cos(r_{qt}) \lambda + \dots$$

that

$$\begin{aligned}\lambda F_2(\lambda) &= -\frac{iy_0^2 t}{2\sigma^2} (\tau_1 + \tau_2 \lambda + \dots) \left(i \sin(r_{qt}) - i \frac{t^2}{2r_{qt}} \cos(r_{qt}) \lambda + \dots \right) \\ &= \frac{y_0^2 t}{2\sigma^2} \tau_1 \sin(r_{qt}) + \frac{y_0^2 t}{2\sigma^2} \left(\tau_2 \sin(r_{qt}) - \tau_1 \frac{t^2}{2r_{qt}} \cos(r_{qt}) \right) \lambda + \dots\end{aligned}$$

Therefore,

$$\alpha_2 = \frac{y_0^2 t}{2\sigma^2} \tau_1 \sin(r_{qt}) > 0 \quad (66)$$

and

$$\tilde{F}_2(0) = \frac{y_0^2 t}{2\sigma^2} \tau_1 \sin(r_{qt}) + \frac{y_0^2 t}{2\sigma^2} \left(\tau_2 \sin(r_{qt}) - \tau_1 \frac{t^2}{2r_{qt}} \cos(r_{qt}) \right). \quad (67)$$

Moreover, (59), (65), and the fact that

$$\frac{\cosh(t\sqrt{\lambda + v_{q,t}}) - 1}{\sqrt{\lambda + v_{q,t}}} = i \frac{1 - \cos(r_{qt})}{r_{qt}} - i \frac{t^3 (2r_{qt} \sin(r_{qt}) + \cos(r_{qt}) - 1)}{2r_{qt}^3} \lambda + \dots$$

give

$$\begin{aligned}\lambda F_3(\lambda) &= -\frac{im^2 q^2 t}{2\sigma^2} (\tau_1 + \tau_2 \lambda + \dots) \left(i \frac{1 - \cos(r_{qt})}{r_{qt}} - i \frac{t^3 (2r_{qt} \sin(r_{qt}) + \cos(r_{qt}) - 1)}{2r_{qt}^3} \lambda + \dots \right) \\ &= \frac{m^2 q^2 t}{2\sigma^2} \tau_1 \frac{1 - \cos(r_{qt})}{r_{qt}} + \frac{m^2 q^2 t}{2\sigma^2} \left[\tau_2 \frac{1 - \cos(r_{qt})}{r_{qt}} - \tau_1 \frac{t^3 (2r_{qt} \sin(r_{qt}) + \cos(r_{qt}) - 1)}{2r_{qt}^3} \right] \lambda + \dots\end{aligned}$$

Hence,

$$\alpha_3 = \frac{m^2 q^2 t}{2\sigma^2} \tau_1 \frac{1 - \cos(r_{qt})}{r_{qt}} > 0 \quad (68)$$

and

$$\tilde{F}_3(0) = \frac{m^2 q^2 t}{2\sigma^2} \left[\tau_2 \frac{1 - \cos(r_{qt})}{r_{qt}} - \tau_1 \frac{t^3 (2r_{qt} \sin(r_{qt}) + \cos(r_{qt}) - 1)}{2r_{qt}^3} \right]. \quad (69)$$

In addition, (60), (65), and the formulas $r_{qt} \cos(r_{qt}) + qt \sin(r_{qt}) = 0$ and

$$\begin{aligned}& \frac{\sinh(t\sqrt{\lambda + v_{q,t}}) - t\sqrt{\lambda + v_{q,t}} \cosh(t\sqrt{\lambda + v_{q,t}})}{\lambda + v_{q,t}} \\ &= -i \frac{t^2 (1 + qt) \sin(r_{qt})}{r_{qt}^2} - i \frac{t^4 \sin(r_{qt}) (1 - qt - r_{qt}^2)}{r_{qt}^4} \lambda + \dots\end{aligned}$$

imply that

$$\lambda F_4(\lambda) = \frac{im^2 q^2 t}{2\sigma^2} (\tau_1 + \tau_2 \lambda + \dots) \left(-i \frac{t^2 (1 + qt) \sin(r_{qt})}{r_{qt}^2} - i \frac{t^4 \sin(r_{qt}) (1 - qt - r_{qt}^2)}{r_{qt}^4} \lambda + \dots \right).$$

It follows that

$$\alpha_4 = \tau_1 \frac{m^2 q^2 t^3 (1 + qt) \sin(r_{qt})}{2\sigma^2 r_{qt}^2} > 0 \quad (70)$$

and

$$\tilde{F}_4(0) = \frac{m^2 q^2 t}{2\sigma^2} \left(\tau_1 \frac{t^4 \sin(r_{qt}) (1 - qt - r_{qt}^2)}{r_{qt}^4} + \tau_2 \frac{t^2 (1 + qt) \sin(r_{qt})}{r_{qt}^2} \right). \quad (71)$$

Finally, we have

$$\frac{4 \sinh^2 \frac{t\sqrt{\lambda+v_{q,t}}}{2} - t\sqrt{\lambda+v_{q,t}} \sinh(t\sqrt{\lambda+v_{q,t}})}{(\lambda+v_{q,t})^{\frac{3}{2}}} = i \frac{t^3 (r_{qt} \sin(r_{qt}) - 4 \sin^2 \frac{r_{qt}}{2})}{r_{qt}^3} \\ + \left[\frac{i3t^5}{2r_{qt}^5} \left(r_{qt} \sin(r_{qt}) - 4 \sin^2 \frac{r_{qt}}{2} \right) + \frac{it^5}{2r_{qt}^3} \left(\frac{3}{r_{qt}} \sin(r_{qt}) - \cos(r_{qt}) \right) \right] \lambda + \dots,$$

and hence

$$\lambda F_5(\lambda) = \frac{im^2 q^3 t}{2\sigma^2} (\tau_1 + \tau_2 \lambda + \dots) \left[i \frac{t^3 (r_{qt} \sin(r_{qt}) - 4 \sin^2 \frac{r_{qt}}{2})}{r_{qt}^3} + D \lambda + \dots \right]$$

where

$$D = \frac{i3t^5}{2r_{qt}^5} \left(r_{qt} \sin(r_{qt}) - 4 \sin^2 \frac{r_{qt}}{2} \right) + \frac{it^5}{2r_{qt}^3} \left(\frac{3}{r_{qt}} \sin(r_{qt}) - \cos(r_{qt}) \right).$$

Using the previous equalities, we obtain

$$\alpha_5 = \frac{m^2 q^3 t^4}{2\sigma^2} \tau_1 \frac{4 \sin^2 \frac{r_{qt}}{2} - r_{qt} \sin(r_{qt})}{r_{qt}^3} \quad (72)$$

and

$$\tilde{F}_5(0) = \frac{m^2 q^3 t^4}{2\sigma^2} \tau_2 \frac{4 \sin^2 \frac{r_{qt}}{2} - r_{qt} \sin(r_{qt})}{r_{qt}^3} \quad (73)$$

$$- \frac{m^2 q^3 t^4}{2\sigma^2} \tau_1 \left[\frac{3t^5}{2r_{qt}^5} \left(r_{qt} \sin(r_{qt}) - 4 \sin^2 \frac{r_{qt}}{2} \right) + \frac{t^5}{2r_{qt}^3} \left(\frac{3}{r_{qt}} \sin(r_{qt}) - \cos(r_{qt}) \right) \right]. \quad (74)$$

Our next goal is to prove that $\alpha_5 > 0$. Indeed, we have

$$4 \sin^2 \frac{r_{qt}}{2} - r_{qt} \sin(r_{qt}) = 2 \sin \frac{r_{qt}}{2} \left(2 \sin \frac{r_{qt}}{2} - r_{qt} \cos \frac{r_{qt}}{2} \right). \quad (75)$$

Since $\frac{\pi}{4} < \frac{r_{qt}}{2} < \frac{\pi}{2}$ and $x < \tan x$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$, inequality $\alpha_5 > 0$ follows from (72) and (75).

We can now apply Theorem 5.2 with the data given by (63) and (64). Recall that we denoted by α the positive number given by

$$\alpha = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \quad (76)$$

where $\alpha_2, \alpha_3, \alpha_4$, and α_5 are defined in (66), (68), (70), and (72), respectively. We also denoted by F the function $F_2 + F_3 + F_4 + F_5$ (see (64)), and put

$$\tilde{F}(0) = \tilde{F}_2(0) + \tilde{F}_3(0) + \tilde{F}_4(0) + \tilde{F}_5(0) \quad (77)$$

where $\tilde{F}_2(0), \tilde{F}_3(0), \tilde{F}_4(0)$, and $\tilde{F}_5(0)$ are given by (67), (69), (71), and (74), respectively. By applying Theorem 5.2 to (57), we see that

$$y^{-\frac{1}{2}} \exp \left\{ (q^2 - v_{q,t}) y \right\} m_t \left(\frac{\sqrt{2}\sigma}{\sqrt{t}} \sqrt{y} \right) = \frac{\sqrt{t}}{\sqrt{2\pi}\sigma} e^{\frac{qt}{2}} \lambda_1^{\frac{1}{2}} e^{\tilde{F}(0)} y^{-\frac{1}{2}} e^{2\sqrt{\alpha}\sqrt{y}} \left(1 + O\left(y^{-\frac{1}{4}}\right) \right) \quad (78)$$

as $y \rightarrow \infty$. Next, replacing y by $y^2 \frac{t}{2\sigma^2}$ in formula (78), we obtain

$$m_t(y) = \frac{\sqrt{t}}{\sqrt{2\pi}\sigma} e^{\frac{qt}{2}} \lambda_1^{\frac{1}{2}} e^{\tilde{F}(0)} \exp \left\{ \frac{\sqrt{2\alpha t}}{\sigma} y \right\} \exp \left\{ -\frac{t(q^2 - v_{q,t})}{2\sigma^2} y^2 \right\} \left(1 + O\left(y^{-\frac{1}{2}}\right) \right) \quad (79)$$

as $y \rightarrow \infty$. Now, it is clear that formula (79) implies Theorem 2.3. Moreover, the following lemma holds:

Lemma 8.2 *The constants E, F , and G in Theorem 2.3 are given by*

$$E = \frac{\sqrt{t}}{\sqrt{2\pi}\sigma} e^{\frac{qt}{2}} \lambda_1^{\frac{1}{2}} e^{\tilde{F}(0)}, \quad F = \sigma^{-1} \sqrt{2\alpha t}, \quad \text{and} \quad G = (2\sigma^2)^{-1} t (q^2 + t^{-2} r_{qt}^2),$$

where the numbers $\lambda_1 > 0$, $\alpha > 0$, and $\tilde{F}(0)$ are defined in (62), (76), and (77), respectively.

This completes the proof of Theorem 2.3.

9 Proof of Theorem 2.1

We will first show that it is possible to apply Theorem 7.1 with $A(y) = y^{-1}m_t(y)e^{Gy^2}$, $k = \sqrt{G + \frac{t}{8}}$, $l = F$, $\zeta(y) = Ey^{-1}$, and $b(y) = y^{-\frac{1}{2}}$, where the constants E , F , and G are such as in Lemma 8.2. It is not hard to see that condition 2 in Theorem 7.1 follows from formula (7). In addition, it is clear that condition 3 with $\gamma = 1$ holds. The validity of condition 1 in Theorem 7.1 can be shown by reasoning as in the proof of Lemma 7.2. Here we use formula (57) instead of formula (39).

It follows from Theorem 7.1 that

$$\begin{aligned} & \int_0^\infty y^{-1}m_t(y) \exp \left\{ - \left(\frac{z^2}{y^2} + \frac{ty^2}{8} \right) \right\} dy \\ &= E \frac{\sqrt{\pi}}{2k} \exp \left\{ \frac{F^2}{16k^2} \right\} k^{\frac{1}{2}} z^{-\frac{1}{2}} \exp \left\{ Fk^{-\frac{1}{2}} \sqrt{z} \right\} e^{-2kz} \left(1 + O \left(z^{-\frac{1}{4}} \right) \right) \end{aligned} \quad (80)$$

as $z \rightarrow \infty$. Replacing z by $\frac{\log x}{\sqrt{2t}}$ in formula (80) and taking into account formula (9) and the equality $k = \frac{\sqrt{8G+t}}{2\sqrt{2}}$, we obtain

$$\begin{aligned} D_t(x_0 e^{\mu t} x) &= \frac{E}{x_0 e^{\mu t}} 2^{-\frac{1}{2}} t^{-\frac{1}{4}} (8G+t)^{-\frac{1}{4}} \exp \left\{ \frac{F^2}{2(8G+t)} \right\} \\ &(\log x)^{-\frac{1}{2}} \exp \left\{ \frac{F\sqrt{2}}{t^{\frac{1}{4}}(8G+t)^{\frac{1}{4}}} \sqrt{\log x} \right\} x^{-\left(\frac{3}{2} + \frac{\sqrt{8G+t}}{2\sqrt{t}}\right)} \left(1 + O \left((\log x)^{-\frac{1}{4}} \right) \right) \end{aligned} \quad (81)$$

as $x \rightarrow \infty$.

Now, it is clear that formula (81) implies Theorem 2.1. In addition, the following lemma holds:

Lemma 9.1 *The constants B_1 , B_2 , and B_3 are given by $B_1 = \frac{E}{x_0 e^{\mu t}} 2^{-\frac{1}{2}} t^{-\frac{1}{4}} (8G+t)^{-\frac{1}{4}} \exp \left\{ \frac{F^2}{2(8G+t)} \right\}$, $B_2 = \frac{F\sqrt{2}}{t^{\frac{1}{4}}(8G+t)^{\frac{1}{4}}}$, and $B_3 = \frac{3}{2} + \frac{\sqrt{8G+t}}{2\sqrt{t}}$, where the numbers E , F , and G are defined in Lemma 8.2.*

It is an interesting fact that Theorem 2.1 with $m = 0$ is a special case of Theorem 2.2. This will be explained below. Recall that in model (1), the volatility process is the absolute value of an Ornstein-Uhlenbeck process. The following explicit representation is valid for the Ornstein-Uhlenbeck process \tilde{Y}_t satisfying the stochastic differential equation $d\tilde{Y}_t = q(m - \tilde{Y}_t)dt + \sigma dZ_t$:

$$\tilde{Y}_t(q, m, \sigma, y_0) = e^{-qt} y_0 + (1 - e^{-qt}) m + \sigma e^{-qt} \int_0^t e^{qu} dZ_u$$

(see, e.g., [16], Proposition 3.8). Therefore,

$$\tilde{Y}_t(q, m, \sigma, y_0) = \tilde{Y}_t(q, 0, \sigma, y_0 + (e^{qt} - 1)m). \quad (82)$$

It is known that squared Ornstein-Uhlenbeck processes are related to CIR-processes. Indeed, it is not hard to see, using the Itô formula, that the squared Ornstein-Uhlenbeck process $T_t = \tilde{Y}_t(q, 0, \sigma, y_0)^2$ satisfies the following stochastic differential equation: $dT_t = (\sigma^2 - 2qT_t)dt + 2\sigma\sqrt{T_t}dZ_t$. Therefore, the uniqueness implies that the process $\tilde{Y}_t(q, 0, \sigma, z_0)^2$ is indistinguishable from the CIR-process $Y_t(\sigma^2, -2q, 2\sigma, z_0^2)$. It follows from (82) that

$$\tilde{Y}_t(q, m, \sigma, y_0)^2 = Y_t(\sigma^2, -2q, 2\sigma, (y_0 + (e^{qt} - 1)m)^2), \quad (83)$$

and hence in the case where $m = 0$, the mixing distribution densities corresponding to the processes on the both sides of (83) coincide. Therefore, all the results concerning the distribution density of the stock price

process in model (2) can be reformulated for model (1) with $m = 0$. For instance, formula (15) becomes

$$\begin{aligned} & \int_0^\infty \exp\{-\lambda y\} y^{-\frac{1}{2}} m_t\left(t^{-\frac{1}{2}} y^{\frac{1}{2}}; q, 0, \sigma, y_0\right) dy \\ &= 2\sqrt{t} \exp\left\{\frac{qt}{2}\right\} \left(\frac{\sqrt{q^2 + 2\sigma^2\lambda}}{\sqrt{q^2 + 2\sigma^2\lambda} \cosh(t\sqrt{q^2 + 2\sigma^2\lambda}) + q \sinh(t\sqrt{q^2 + 2\sigma^2\lambda})} \right)^{\frac{1}{2}} \\ & \quad \exp\left\{-\frac{y_0^2 \lambda \sinh(t\sqrt{q^2 + 2\sigma^2\lambda})}{\sqrt{q^2 + 2\sigma^2\lambda} \cosh(t\sqrt{q^2 + 2\sigma^2\lambda}) + q \sinh(t\sqrt{q^2 + 2\sigma^2\lambda})}\right\}. \end{aligned}$$

Summarizing what was said above, we see that Theorem 2.1 with $m = 0$ follows from Theorem 2.2 and formula (83).

The next statement concerns the moment explosion problem for the Stein-Stein model.

Lemma 9.2 *Let $q \geq 0$, $m \geq 0$, $\sigma > 0$, and $p \in \mathbb{R}$. Then, the following statement is true for the moment l_p of the stock price distribution density D_t in model (1):*

$$l_p < \infty \iff \frac{1}{2} - \frac{\sqrt{8C+t}}{2\sqrt{t}} < p < \frac{1}{2} + \frac{\sqrt{8C+t}}{2\sqrt{t}},$$

where $C = \frac{1}{2\sigma^2} (tq^2 + t^{-1}r_{tq}^2)$.

It is not hard to see that Lemma 9.2 follows from Lemma 7.4 and formula (83).

10 Proof of Theorem 3.1

Let $\bar{K} > 0$, and let f and g be positive functions on the interval $[\bar{K}, \infty)$. We will write $f(x) \approx g(x)$, $x \rightarrow \infty$, if there exist constants $c_1 > 0$, $c_2 > 0$, and $K_0 > \bar{K}$ such that the inequalities $c_1 g(x) \leq f(x) \leq c_2 g(x)$ hold for all $x > K_0$. The following notation will be used below: $\hat{V}_0(k) = V_0(K)$ (see (11) for the definition of V_0).

The next lemma was obtained in [13].

Lemma 10.1 *Suppose that there exist positive increasing continuous functions ψ and ϕ such that $\lim_{k \rightarrow \infty} \psi(k) = \lim_{k \rightarrow \infty} \phi(k) = \infty$ and*

$$\hat{V}_0(k) \approx \frac{\psi(k)}{\phi(k)} \exp\left\{-\frac{\phi(k)^2}{2}\right\} \quad (84)$$

as $K \rightarrow \infty$. Then the following asymptotic formula holds:

$$\hat{I}(k) = \frac{1}{\sqrt{T}} \left(\sqrt{2k + \phi(k)^2} - \phi(k) \right) + O\left(\frac{\psi(k)}{\phi(k)}\right)$$

as $K \rightarrow \infty$.

For the model in (2), let ψ be a function such as in the formulation of Lemma 10.1. Our next goal is to find a function ϕ for which formula (84) holds. It follows from (12) that

$$\hat{V}_0(k) = e^{-rT} (x_0 e^{rT})^2 \left[\int_{e^k}^\infty y D_T(x_0 e^{rT} y) dy - e^k \int_{e^k}^\infty D_T(x_0 e^{rT} y) dy \right]. \quad (85)$$

Now it is not hard to see that (6) and (85) imply

$$\hat{V}_0(k) \approx k^{-\frac{3}{4} + \frac{a}{c^2}} e^{A_2 \sqrt{k}} e^{-(A_3 - 2)k}, \quad k \rightarrow \infty. \quad (86)$$

Put

$$\phi(k) = \sqrt{(2A_3 - 4)k - 2A_2 \sqrt{k} + \left(\frac{1}{2} - \frac{2a}{c^2}\right) \log k - 2 \log \psi(k)}.$$

Then we have $\phi(k) \approx \sqrt{k}$, and (86) shows that condition (84) in Lemma 10.1 holds. Applying this lemma and the mean value theorem, we see that

$$\hat{I}(k) = \frac{\sqrt{2}}{\sqrt{T}} \left[\sqrt{(A_3 - 1)k - A_2\sqrt{k} + \left(\frac{1}{4} - \frac{a}{c^2}\right)\log k} - \sqrt{(A_3 - 2)k - A_2\sqrt{k} + \left(\frac{1}{4} - \frac{a}{c^2}\right)\log k} \right] + O\left(\frac{\psi(k)}{\sqrt{k}}\right), \quad k \rightarrow \infty.$$

Next using the fact that $\sqrt{1-h} = 1 - \frac{1}{2}h + O(h^2)$ as $h \downarrow 0$, we obtain (13). The proof of (14) is similar. Here we use (5) instead of (6).

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